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Likelihood Ratio Test for the Equivalence of Two Autoregressive Moving-Average Time Series

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PREFACE

The work described in this report was initiated under NUWC Division Newport Project No. C600068, "VLF (Very Low-Frequency) Direction-Finding Project," principal investigator Norman L. Owsley (formerly of Code 2123). It was completed under NUWC Division Newport Project No. A600359, "Tactical Towed Array Localization Project," principal investigator John P. Ianniello (Code 2123). The sponsoring activity is the Office of Naval Research, program managers Kenneth Dial and Nancy Harned (ONR 321).

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13. ABSTRACT (Maximum 200 words)

To passively detect quiet sources, future sonar systems will require more sensors, which may contribute to operator overload. The methods described in this report have the potential to automatically determine if two sonar tracks, for example, correspond to the same source, thereby improving operator performance.

Specifically, a likelihood ratio test for the equivalence of two autoregressive moving-average (ARMA) time series is derived. This test investigates the structural characteristics of the two time series through the ARMA parameters. Four cases of this test are presented for examining the ARMA parameters, series means, and/or innovations variances. The autoregressive (AR) time series is treated separately, not only because AR parameters are easier to estimate, but also because many time series can be characterized by an AR process. Monte Carlo analysis has shown that the likelihood ratio test has a good fit to the chi-square distribution, with degrees of freedom equal to the number of parameters being tested.

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LIKELIHOOD RATIO TEST FOR THE EQUIVALENCE OF TWO AUTOREGRESSIVE MOVING-AVERAGE TIME SERIES

1. INTRODUCTION

In many areas of spatial and temporal signal processing, the analysis of the relationship between two (or more) time series is of interest; time delay estimation is one example. Crosscorrelation analysis can provide insight into a linear relationship between the signals; however, it does not compare their structural details. For future sonar systems, the procedures described in this report have the potential to improve sonar operator performance by automatically determining if two signals or sonar tracks are associated with the same source.

Specifically, two independent autoregressive moving-average (ARMA) time series are analyzed to determine whether or not the series may be characterized by the same ARMA model. If the same model can be successfully applied to each series, the implication is that the series may have the same parent process; further analysis of the series is then warranted.

To test the equivalence of two ARMA time series models, a likelihood ratio statistic is derived. The test statistic $-2 \cdot \log T$, where T is the likelihood ratio statistic, has an asymptotic chi-square distribution with r degrees of freedom. This value is defined as the number of model parameters in series 1 that are being tested for equality to the same model parameters in series 2.

In section 2, the likelihood ratio test is derived to measure the equivalence of two autoregressive (AR) processes with Gaussian innovations. Because the AR model parameters are easier to estimate and because many time series can be characterized as AR processes, this special case is treated separately.

Section 3 presents the likelihood ratio for the equivalence of two ARMA processes with Gaussian innovations. Results from simulated data are described in section 4. Section 5 discusses a general form of the test, as well as provides the conclusions.

2. DERIVATION OF THE LIKELIHOOD RATIO FOR AR PROCESSES

Let $x_1,\ x_2,\ \ldots\ ,\ x_{nx}$ and $y_1,\ y_2,\ \ldots\ ,\ y_{ny}$ be realizations of the autoregressions with respective means μ_x and μ_y defined by

$$x_t \;=\; \mu_x + \sum_{k=1}^{px} \alpha_k\,(x_{t-k} - \mu_x) + u_t$$

and

$$y_t \;=\; \mu_y + \sum_{k=1}^{py} \beta_k\,(y_{t-k} - \mu_y) + v_t\ , \qquad (1)$$

where $\alpha_1,\ \alpha_2,\ \ldots\ ,\ \alpha_{px}$ and $\beta_1,\ \beta_2,\ \ldots\ ,\ \beta_{py}$ are the parameters for the AR processes x_t and y_t with model orders px and py, respectively. The values u_t and v_t are independent Gaussian random variables with

$$\begin{aligned}
E[u_t] &= \mu_x\ , \\
E[v_t] &= \mu_t\ , \\
E[u_s u_t] &= \sigma_u^2\,\delta(s-t)\ , \\
E[v_s v_t] &= \sigma_v^2\,\delta(s-t)\ , \\
E[u_s v_t] &= 0 \text{ for all } s \text{ and } t\ ,
\end{aligned}$$

where $E[\cdot]$ denotes the statistical expectation operator and $\delta(\cdot)$ is the Kronecker delta function.

For the present discussion, it is of interest to know whether x_t $(t = 1,\ 2,\ \ldots\ ,\ nx)$ and y_t $(t = 1,\ 2,\ \ldots\ ,\ ny)$ are realizations of the same AR process. Since the objective is to test whether x_t and y_t have the same parent AR process, px and py are set equal to p.

Because u_t and v_t are assumed to be independent Gaussian random variables, the likelihood function of $\{u_t\}_{p+1}^{nx}$ and $\{v_t\}_{p+1}^{ny}$ conditioned on $\{x_t\}_1^p$ and $\{y_t\}_1^p$ is

$$L(\mathbf{x},\ \mathbf{y};\ \boldsymbol{\alpha},\ \boldsymbol{\beta},\ \sigma_u^2,\ \sigma_v^2,\ \mu_x,\ \mu_y) \;=\; L(\mathbf{x};\ \boldsymbol{\alpha},\ \sigma_u^2,\ \mu_x) * L(\ \mathbf{y};\ \boldsymbol{\beta},\ \sigma_v^2,\ \mu_y)\ , \qquad (2)$$

where

$$L(\mathbf{x}\ ;\ \boldsymbol{\alpha},\ \sigma_u^2,\ \mu_x) \ = \ (2\pi\sigma_u^2)^{-(nx-p)/2}\exp(-\sum_{t=p+1}^{nx}(x_t-\mu_x-\sum_{k=1}^{p}\alpha_k(x_{t-k}-\mu_x))^2/2\sigma_u^2) \tag{3}$$

and

$$L(\mathbf{y}\ ;\ \boldsymbol{\beta},\sigma_v^2,\ \mu_y) \ = \ (2\pi\sigma_v^2)^{-(ny-p)/2}\exp(-\sum_{t=p+1}^{ny}(y_t-\mu_y-\sum_{k=1}^{p}\beta_k(y_{t-k}-\mu_y))^2/2\sigma_v^2)\ . \tag{4}$$

The vectors $\mathbf{x}$, $\mathbf{y}$, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$ are defined by

$$\begin{aligned} \mathbf{x} &= (x_{p+1}, x_{p+2}, ..., x_{nx})^T, \\ \mathbf{y} &= (y_{p+1}, y_{p+2}, ..., y_{ny})^T, \\ \boldsymbol{\alpha} &= (\alpha_1, \alpha_2, ..., \alpha_p)^T, \end{aligned}$$

and

$$\boldsymbol{\beta} = (\beta_1, \beta_2, .., \beta_p)^T,$$

where the superscript T denotes the transpose of a matrix.

The parameters examined and the form of the four likelihood ratio tests are as follows:

1. A test for equal AR parameters ($\alpha_k = \beta_k$, $k = 1, 2, \ldots, p$) and innovations variances ($\sigma_x^2 = \sigma_y^2$):

$$T = \frac{\max\limits_{\boldsymbol{\alpha}=\boldsymbol{\beta}\ ,\ \sigma_u^2=\sigma_v^2\ ,\ \mu_x\ ,\ \mu_y} L(\mathbf{x},\mathbf{y}\ ;\ \boldsymbol{\alpha},\boldsymbol{\beta},\sigma_u^2,\sigma_v^2,\ \mu_x,\ \mu_y)}{\max\limits_{\boldsymbol{\alpha}\ ,\ \boldsymbol{\beta}\ ,\ \sigma_u^2\ ,\ \sigma_v^2\ ,\ \mu_x\ ,\ \mu_y} L(\mathbf{x},\mathbf{y}\ ;\ \boldsymbol{\alpha},\boldsymbol{\beta},\sigma_u^2,\sigma_v^2,\ \mu_x,\ \mu_y)}\ . \tag{5}$$

2. A test for equal AR parameters ($\alpha_k = \beta_k$, $k = 1, 2, \ldots, p$), innovations variances $\left(\sigma_x^2 = \sigma_y^2\right)$, and process means $\left(\mu_x = \mu_y\right)$:

$$T = \frac{\max\limits_{\boldsymbol{\alpha}=\boldsymbol{\beta},\ \mu_x=\mu_y,\ \sigma_x^2=\sigma_y^2} L(\mathbf{x},\ \mathbf{y};\ \boldsymbol{\alpha},\ \boldsymbol{\beta},\ \sigma_u^2,\ \sigma_v^2,\ \mu_x,\ \mu_y)}{\max\limits_{\boldsymbol{\alpha},\ \boldsymbol{\beta},\ \sigma_u^2,\ \sigma_v^2,\mu_x,\ \mu_y} L(\mathbf{x},\ \mathbf{y};\ \boldsymbol{\alpha},\ \boldsymbol{\beta},\ \sigma_u^2,\ \sigma_v^2,\mu_x,\ \mu_y)}\ . \tag{6}$$

3. A test for equal AR parameters ($\alpha_k = \beta_k$, $k = 1, 2, \dots, p$):

$$T = \frac{\displaystyle\max_{\boldsymbol{\alpha} = \boldsymbol{\beta},\, \sigma_u^2,\, \sigma_v^2,\, \mu_x,\, \mu_y} L(\mathbf{x}, \mathbf{y}; \boldsymbol{\alpha}, \boldsymbol{\beta}, \sigma_u^2, \sigma_v^2, \mu_x, \mu_y)}{\displaystyle\max_{\boldsymbol{\alpha},\, \boldsymbol{\beta},\, \sigma_u^2,\, \sigma_v^2,\, \mu_x,\, \mu_y} L(\mathbf{x}, \mathbf{y}; \boldsymbol{\alpha}, \boldsymbol{\beta}, \sigma_u^2, \sigma_v^2)} . \tag{7}$$

4. A test for equal AR parameters ($\alpha_k = \beta_k$, $k = 1, 2, \dots, p$) and process means ($\mu_x = \mu_y$):

$$T = \frac{\displaystyle\max_{\boldsymbol{\alpha} = \boldsymbol{\beta},\, \sigma_u^2,\, \sigma_v^2,\, \mu_x = \mu_y} L(\mathbf{x}, \mathbf{y}; \boldsymbol{\alpha}, \boldsymbol{\beta}, \sigma_u^2, \sigma_v^2, \mu_x, \mu_y)}{\displaystyle\max_{\boldsymbol{\alpha},\, \boldsymbol{\beta},\, \sigma_u^2,\, \sigma_v^2,\, \mu_x,\, \mu_y} L(\mathbf{x}, \mathbf{y}; \boldsymbol{\alpha}, \boldsymbol{\beta}, \sigma_u^2, \sigma_v^2, \mu_x, \mu_y)} . \tag{8}$$

To evaluate T, a value of the model order p must be provided. Using the Akaike information criteria (AIC) or one of the other selection procedures available[1–3] allows model orders px, py, and pxy to be obtained for x_t, y_t, and the pooled data of x_t and y_t respectively. Three possible ways to select p are

(a) p = minimum of px and py,

(b) p = maximum of px and py, or

(c) $p = pxy$.

Simulation experiments have shown that selecting p using (a) or (b) results in a liberal test that rejects the null hypothesis when it is true more often than is predicted by the probability distribution of the test statistic. Using (c), or $p = pxy$ (the selected model order of the pooled data), yields a test statistic that has a good fit to the chi-square distribution with the appropriate degrees of freedom.

2.1 TEST FOR EQUAL AR PARAMETERS AND INNOVATIONS VARIANCES

This section examines whether $\alpha_k = \beta_k$ ($k = 1, 2, \dots, p$) and $\sigma_x^2 = \sigma_y^2$. Since this test is not concerned with the means μ_x and μ_y, it is assumed that the sample means $\overline{x} = (1/nx)\sum_{t=1}^{nx} x_t$ and $\overline{y} = (1/ny)\sum_{t=1}^{ny} \mathrm{y}_t$ have been subtracted from the realizations x_t (where $t = 1, 2, \dots, nx$) and y_t (where $t = 1, 2, \dots, ny$), respectively.

The likelihood ratio test statistic to determine whether the two realizations can be

characterized by the same AR process is given by equation (5). The maximum of the denominator is obtained at the values of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ that satisfy

$$\mathbf{A}\boldsymbol{\alpha} = \mathbf{a} \tag{9}$$

and

$$\mathbf{B}\boldsymbol{\beta} = \mathbf{b} \;, \tag{10}$$

where the ij-th element of $\mathbf{A}$ and $\mathbf{B}$ are given by

$$A_{ij} = \sum_{t=p+1}^{nx} x_{t-i}x_{t-j}$$

and

$$B_{ij} = \sum_{t=p+1}^{ny} y_{t-i}y_{t-j} \;,$$

and the i-th entry in $\mathbf{a}$ and $\mathbf{b}$ are given by

$$a_i = \sum_{t=p+1}^{nx} x_t x_{t-i}$$

and

$$b_i = \sum_{t=p+1}^{ny} y_t y_{t-i} \;.$$

Using the solutions of equations (9) and (10), denoted as $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$, estimates of σ_u^2 and σ_v^2 are obtained from

$$\hat{\sigma}_u^2 = (1/(nx-p)) \sum_{t=p+1}^{nx} (x_t - \sum_{k=1}^{p} \hat{\alpha}_k \; x_{t-k})^2 \tag{11}$$

and

$$\hat{\sigma}_v^2 = (1/(ny-p)) \sum_{t=p+1}^{ny} (y_t - \sum_{k=1}^{p} \hat{\beta}_k \; y_{t-k})^2. \tag{12}$$

The maximum of the numerator is obtained at the value of $\boldsymbol{\alpha}(\boldsymbol{\beta})$ that satisfies

$$\mathbf{C}\boldsymbol{\alpha} = \mathbf{c} \; , \tag{13}$$

where

$$C_{ij} \;=\; \sum_{t=p+1}^{nx} x_{t-i}\, x_{t-j} + \sum_{t=p+1}^{ny} y_{t-i}\, y_{t-j}$$

and

$$c_i \;=\; \sum_{t=p+1}^{nx} x_t\, x_{t-i} + \sum_{t=p+1}^{ny} y_t\, y_{t-i} \; .$$

The estimate of σ^2 is

$$\tilde{\sigma}^2 \;=\; \left[\sum_{t=p+1}^{nx} (x_t - \sum_{k=1}^{p} \tilde{\alpha}_k\, x_{t-k})^2 + \sum_{t=p+1}^{ny} (y_t - \sum_{k=1}^{p} \tilde{\alpha}_k\, y_{t-k})^2 \right] /(nx + ny - 2p) \; , \tag{14}$$

where $\tilde{\boldsymbol{\alpha}}$ is the solution of equation (13).

Substituting $\hat{\boldsymbol{\alpha}}$, $\hat{\boldsymbol{\beta}}$, $\hat{\sigma}_u^2$, $\hat{\sigma}_v^2$, $\tilde{\alpha}$, and $\tilde{\sigma}^2$ into equations (2) and (5) yields

$$T \;=\; \frac{(\tilde{\sigma}^2)^{-(nx+ny-2p)/2}}{(\hat{\sigma}_u^2)^{-(nx-p)/2} (\hat{\sigma}_v^2)^{-(ny-p)/2}} \; . \tag{15}$$

Taking $-2 \cdot \log T$ of equation (15) yields

$$-\,2 \cdot \log T = (nx + ny - 2p) \log \tilde{\sigma}^2 - (nx - p) \log \hat{\sigma}_x^2 - (ny - p) \log \hat{\sigma}_y^2 \; . \tag{16}$$

Under the null hypothesis, $\boldsymbol{\alpha} = \boldsymbol{\beta}$ and $\sigma_u^2 = \sigma_v^2$, $-2 \cdot \log T$ has an asymptotic chi-square distribution with $p + 1$ degrees of freedom.

2.2 TEST FOR EQUAL AR PARAMETERS, PROCESS MEANS, AND INNOVATIONS VARIANCES

The previous section derived the test statistic for zero-mean processes. In this section, the test is extended to include the case where x_t and y_t have means μ_x and μ_y, respectively. Including the means in equation (1) results in

$$x_t = \mu_x + \sum_{k=1}^{p} \alpha_k \left(x_{t-k} - \mu_x \right) + u_t \ ,$$

$$y_t = \mu_y + \sum_{k=1}^{p} \beta_k \left(y_{t-k} - \mu_y \right) + v_t \ . \tag{17}$$

If equation (17) is used, expression (2) becomes

$$\begin{aligned} L(\mathbf{x}, \mathbf{y}; \boldsymbol{\alpha}, \boldsymbol{\beta}, \sigma_x^2, \sigma_y^2, \mu_x, \mu_y) &= (2\pi\sigma_x^2)^{-(nx-p)/2} (2\pi\sigma_y^2)^{-(ny-p)/2} \\ &\cdot \exp\left[- \sum_{t=p+1}^{nx} \left((x_t - \mu_x) - \sum_{k=1}^{p} \alpha_k (x_{t-k} - \mu_x) \right)^2 / 2\,\sigma_x^2 \right] \\ &\cdot \exp\left[- \sum_{t=p+1}^{ny} \left((y_t - \mu_y) - \sum_{k=1}^{p} \beta_k (y_{t-k} - \mu_y) \right)^2 / 2\,\sigma_y^2 \right] , \end{aligned} \tag{18}$$

and the test statistic is equation (6).

The maximum of the denominator in equation (6) is obtained by solving equations (9) and (10), where the ij-th element of $\mathbf{A}$ and $\mathbf{B}$ are given by

$$A_{ij} = \sum_{t=p+1}^{nx} \tilde{x}_{t-i}\, \tilde{x}_{t-j}$$

and

$$B_{ij} = \sum_{t=p+1}^{ny} \tilde{y}_{t-i}\, \tilde{y}_{t-j} \ ,$$

and the i-th entry of $\mathbf{a}$ and $\mathbf{b}$ are given by

$$a_i = \sum_{t=p+1}^{nx} \tilde{x}_t \, \tilde{x}_{t-i}$$

and

$$b_i = \sum_{t=p+1}^{ny} \tilde{y}_t \, \tilde{y}_{t-i} \; .$$

In these expressions, $\tilde{x}_t = x_t - \bar{x}$ and $\tilde{y}_t = y_t - \bar{y}$, where $\bar{x} = \sum_{t=1}^{nx} x_t \,/\, nx$ and $\bar{y} = \sum_{t=1}^{ny} y_t \,/\, ny$.

As in equations (11) and (12), the estimates of σ_x^2 and σ_y^2 are given by

$$\hat{\sigma}_u^2 = (1/(nx-p)) \sum_{t=p+1}^{nx} (\tilde{x}_t - \sum_{k=1}^{p} \hat{\alpha}_k \, \tilde{x}_{t-k})^2 \tag{19}$$

and

$$\hat{\sigma}_v^2 = (1/(ny-p)) \sum_{t=p+1}^{ny} (\tilde{y}_t - \sum_{k=1}^{p} \hat{\beta}_k \, \tilde{y}_{t-k})^2 . \tag{20}$$

The maximum of the numerator is obtained by solving equation (13), where the ij-th entry of $\mathbf{C}$ is

$$C_{ij} = \sum_{t=p+1}^{nx} \tilde{x}_{t-i}\tilde{x}_{t-j} + \sum_{t=p+1}^{ny} \tilde{y}_{t-i}\tilde{y}_{t-j} \; ,$$

and the i-th entry of $\mathbf{c}$ is

$$c_i = \sum_{t=p+1}^{nx} \tilde{x}_t\tilde{x}_{t-i} + \sum_{t=p+1}^{ny} \tilde{y}_t\tilde{y}_{t-i} \; .$$

In these expressions, $\tilde{x}_t = x_t - \hat{\mu}$ and $\tilde{y}_t = y_t - \hat{\mu}$, where $\hat{\mu} = (nx\,\bar{x} + ny\,\bar{y})/(nx+ny)$.

Equation (14) now becomes

$$\tilde{\sigma}^2 = \left[\sum_{t=p+1}^{nx} \left(\tilde{x}_t - \sum_{k=1}^{p} \hat{\alpha}_k \tilde{x}_{t-k} \right)^2 + \sum_{t=p+1}^{ny} \left(\tilde{y}_t - \sum_{k=1}^{p} \hat{\alpha}_k \tilde{y}_{t-k} \right)^2 \right] / (nx + ny - 2p). \quad (21)$$

The test statistic of equation (16) can be evaluated from equations (19), (20), and (21). Under the null hypothesis $\boldsymbol{\alpha} = \boldsymbol{\beta}$, $\sigma_x^2 = \sigma_y^2$, $\mu_x = \mu_y$, and expression (16) has a chi-square distribution with $p + 2$ degrees of freedom.

2.3 TEST FOR EQUAL AR PARAMETERS

In many applications, there is interest only in the AR structure of the process, not in the means (μ_x and μ_y) and the innovations variances (σ_u^2 and σ_v^2).

The denominator of test statistic (7) is maximized by using equations (9)-(12). However, the numerator is maximized by solving for $\boldsymbol{\alpha}, \sigma_u^2$, and σ_v^2 in the implicit system of equations:

$$\mathbf{D}\boldsymbol{\alpha} = \mathbf{d} \, , \quad (22)$$

$$\sigma_u^2 = 1/(nx - p) \sum_{t=p+1}^{nx} \left(x_t - \sum_{k=1}^{p} \alpha_k x_{t-k} \right)^2 , \quad (23)$$

$$\sigma_v^2 = 1/(ny - p) \sum_{t=p+1}^{ny} \left(y_t - \sum_{k=1}^{p} \alpha_k y_{t-k} \right)^2 , \quad (24)$$

where the ij-th entry of $\mathbf{D}$ is

$$D_{ij} = 1/\sigma_u^2 \sum_{t=p+1}^{nx} x_{t-i} x_{t-j} + 1/\sigma_v^2 \sum_{t=p+1}^{ny} y_{t-i} y_{t-j}$$

and the i-th entry of $\mathbf{d}$ is

$$d_i = 1/\sigma_u^2 \sum_{t=p+1}^{nx} x_t x_{t-j} + 1/\sigma_v^2 \sum_{t=p+1}^{ny} y_t y_{t-j} \, .$$

These equations may be solved using a fixed-point algorithm. The procedure begins by estimating $\boldsymbol{\alpha}$ using equation (22); values for σ_u^2 and σ_v^2 are those obtained from maximizing the denominator of equation (7). With the estimate for $\boldsymbol{\alpha}$, new estimates are obtained for σ_u^2 and σ_v^2 from equations (23) and (24). This procedure is repeated until the change in value of the likelihood function is less than a user-selected convergence criteria. Typically, the algorithm converges in less than five iterations.

The complete likelihood function must be used to evaluate T because the terms involving the exponential do not cancel. Taking $-2 \cdot \log T$ yields

$$\begin{aligned} -2 \cdot \log T &= (nx-p)\log\tilde{\sigma}_u^2 + (ny-p)\log\tilde{\sigma}_v^2 \\ &\quad +(1/\tilde{\sigma}_u^2)\sum_{t=p+1}^{nx}\left(x_t - \sum_{k=1}^{p}\tilde{\alpha}_k x_{t-k}\right)^2 + (1/\tilde{\sigma}_v^2)\sum_{t=p+1}^{ny}\left(y_t - \sum_{k=1}^{p}\tilde{\alpha} y_{t-k}\right)^2 \\ &\quad -(nx-p)\log\hat{\sigma}_u^2 - (ny-p)\log\hat{\sigma}_v^2 - (nx-p) - (ny-p) \end{aligned} \tag{25}$$

or

$$\begin{aligned} -2 \cdot \log T &= (nx-p)\log\left(\tilde{\sigma}_u^2/\hat{\sigma}_u^2\right) + (ny-p)\log\left(\tilde{\sigma}_v^2/\hat{\sigma}_v^2\right) \\ &\quad +(1/\tilde{\sigma}_u^2)\sum_{t=p+1}^{nx}\left(x_t - \sum_{k=1}^{p}\tilde{\alpha}_k x_{t-k}\right)^2 + (1/\tilde{\sigma}_v^2)\sum_{t=p+1}^{ny}\left(y_t - \sum_{k=1}^{p}\tilde{\alpha} y_{t-k}\right)^2 \\ &\quad -(nx+ny-2p), \end{aligned} \tag{26}$$

where $\tilde{\boldsymbol{\alpha}}$, $\tilde{\sigma}_u^2$, $\tilde{\sigma}_v^2$ maximize the numerator and $\hat{\boldsymbol{\alpha}}$, $\hat{\boldsymbol{\beta}}$, $\hat{\sigma}_u^2$, $\hat{\sigma}_v^2$ maximize the denominator of equation (7), respectively. As before, $-2 \cdot \log T$ has a chi-square distribution, but with p degrees of freedom.

2.4 TEST FOR EQUAL AR PARAMETERS AND PROCESS MEANS

For completeness, the case for equal AR parameters and process means is examined. Now the denominator of test (8) is maximized by using equations (9) through (12),

where x_t and y_t are replaced by $x_t - \bar{x}$ and $y_t - \bar{y}$, respectively ($\bar{x} = (1/nx)\sum_{t=1}^{nx} x_t$ and $\bar{y} = (1/ny)\sum_{t=1}^{ny} y_t$). The numerator is maximized by equations (22) through (24), where x_t and y_t are replaced by $x_t - \hat{\mu}$ and $y_t - \hat{\mu}$, respectively ($\hat{\mu} = (nx \cdot \bar{x} + ny \cdot \bar{y})/(nx + ny)$). The expression for the test statistic is given by equation (26). This value, $-2 \cdot \log T$, has a chi-square distribution with $p + 1$ degrees of freedom if $\boldsymbol{\alpha} = \boldsymbol{\beta}$ and $\mu_x = \mu_y$.

3. LIKELIHOOD RATIO TESTS FOR ARMA PROCESSES

This section extends the previous results to ARMA processes. Let these processes x_t and y_t be defined by

$$x_t = \mu_x + \sum_{k=1}^{p} \alpha_k \cdot (x_{t-k} - \mu_x) + u_t + \sum_{k=1}^{q} \theta_k u_{t-k} \tag{27}$$

and

$$y_t = \mu_y + \sum_{k=1}^{p} \beta_k \cdot (y_{t-k} - \mu_y) + v_t + \sum_{k=1}^{q} \phi_k v_{t-k} \ , \tag{28}$$

where u_t and v_t are defined as in equation (1). As for the AR models, the ARMA model orders p and q are the same for both processes x_t and y_t.

The Gaussian likelihood of x_t and y_t, written in terms of conditional probabilities, is

$$L(\mathbf{x}, \mathbf{y}; \boldsymbol{\alpha}, \boldsymbol{\theta}, \boldsymbol{\beta}, \boldsymbol{\phi}, \boldsymbol{\sigma}_u^2, \boldsymbol{\sigma}_v^2, \mu_x, \mu_y) = \prod_{t=1}^{nx} f_x(x_t|\tilde{\mathbf{x}}_{t-1}, \boldsymbol{\alpha}, \boldsymbol{\theta}, \boldsymbol{\sigma}_u^2, \boldsymbol{\mu}_x) \prod_{t=1}^{ny} f_y(y_t|\tilde{\mathbf{y}}_{t-1}, \boldsymbol{\beta}, \boldsymbol{\phi}, \boldsymbol{\sigma}_y^2, \boldsymbol{\mu}_y), \tag{29}$$

where

$$\begin{aligned}
\boldsymbol{\alpha} &= (\alpha_1, \alpha_2, ..., \alpha_p)^T \ , \\
\boldsymbol{\theta} &= (\theta_1, \theta_2, ..., \theta_q)^T \ , \\
\boldsymbol{\beta} &= (\beta_1, \beta_2, ..., \beta_p)^T \ , \\
\boldsymbol{\phi} &= (\phi_1, \phi_2, ..., \phi_q)^T \ , \\
\tilde{\mathbf{x}}_{t-1} &= (x_1, x_2, ..., x_{t-1})^T \ , \\
\tilde{\mathbf{y}}_{t-1} &= (y_1, y_2, ..., y_{t-1})^T \ , \\
f_x(x_1|\tilde{\mathbf{x}}_0, \boldsymbol{\alpha}, \boldsymbol{\theta}, \sigma_u^2, \mu_x) &= f_x(x_1|\boldsymbol{\alpha}, \boldsymbol{\theta}, \sigma_u^2, \mu_x) \ , \\
f_y(y_1|\tilde{\mathbf{y}}_0, \boldsymbol{\beta}, \boldsymbol{\phi}, \sigma_v^2, \mu_y) &= f_y(y_1|\boldsymbol{\beta}, \boldsymbol{\phi}, \sigma_v^2, \mu_y) \ , \\
f_x(x_t|\tilde{\mathbf{x}}_{t-1}, \boldsymbol{\alpha}, \boldsymbol{\theta}, \sigma_u^2, \mu_x) &= (2\pi\sigma_u^2 a_t)^{-1/2} \exp(-(x_t - (\mu_x + x(t|t-1)))^2 / 2\sigma_u^2 a_t) \ , \\
f_y(y_t|\tilde{\mathbf{y}}_{t-1}, \boldsymbol{\beta}, \boldsymbol{\phi}, \sigma_v^2, \mu_y) &= (2\pi\sigma_v^2 b_t)^{-1/2} \exp(-(y_t - (\mu_y + y(t|t-1)))^2 / 2\sigma_v^2 b_t) \ , \\
x(t|t-1) &= E[x_t - \mu_x|\tilde{\mathbf{x}}_{t-1}, \boldsymbol{\alpha}, \boldsymbol{\theta}, \sigma_u^2, \mu_x] \ , \\
y(t|t-1) &= E[y_t - \mu_y|\tilde{\mathbf{y}}_{t-1}, \boldsymbol{\beta}, \boldsymbol{\phi}, \sigma_v^2, \mu_y] \ ,
\end{aligned}$$

$$\sigma_u^2 a_t = Var[x_t|\tilde{\mathbf{x}}_{t-1}, \boldsymbol{\alpha}, \boldsymbol{\theta}, \sigma_u^2, \mu_x] ,$$
$$\sigma_v^2 b_t = Var[y_t|\tilde{\mathbf{y}}_{t-1}, \boldsymbol{\beta}, \boldsymbol{\phi}, \sigma_v^2, \mu_y] .$$

$Var[\cdot]$ denotes the variance of the random variable in the brackets. Substituting $f_x(\cdot)$ and $f_y(\cdot)$ in equation (29) and taking the natural logarithm yields the log likelihood of x_t and y_t. When the constant terms involving 2π are ignored, the log likelihood is given by

$$-2 \cdot \log\ L(\mathbf{x}, \mathbf{y}; \boldsymbol{\alpha}, \boldsymbol{\theta}, \boldsymbol{\beta}, \boldsymbol{\phi}, \boldsymbol{\sigma}_u^2, \boldsymbol{\sigma}_v^2, \boldsymbol{\mu}_x, \boldsymbol{\mu}_y) =$$

$$-nx \cdot \log\ \sigma_u^2 - \sum_{t=1}^{nx} \log\ a_t - \sum_{t=1}^{nx} (x_t - (\mu_x + x(t|t-1)))^2 / \sigma_u^2\ a_t$$

$$-ny \cdot \log\ \sigma_v^2 - \sum_{t=1}^{ny} \log\ b_t - \sum_{t=1}^{ny} (y_t - (\mu_y + y(t|t-1)))^2 / \sigma_v^2\ b_t\ . \tag{30}$$

The log likelihood can be evaluated with a Kalman filter (see appendix) algorithm and can be maximized with respect to $\boldsymbol{\alpha}$, $\boldsymbol{\theta}$, $\boldsymbol{\beta}$, $\boldsymbol{\phi}$, σ_u^2, σ_v^2, μ_x, and μ_y via a nonlinear optimization procedure. The quantities a_t and b_t are calculated as part of the Kalman filter evaluation of the log likelihood. As in section 2, four likelihood ratio tests are presented for the equivalence of ARMA processes. The only difference from the AR tests is the inclusion of the moving-average parameters.

3.1 TEST FOR EQUAL ARMA PARAMETERS AND INNOVATIONS VARIANCES

The test is for $\boldsymbol{\alpha} = \boldsymbol{\beta}$, $\boldsymbol{\theta} = \boldsymbol{\phi}$, and $\sigma_u^2 = \sigma_v^2$. As in the AR test, the sample means $\bar{x}$ and $\bar{y}$ are removed from x_t and y_t, respectively. For this case, the likelihood ratio test is

$$T = \frac{\max\limits_{\boldsymbol{\alpha} = \boldsymbol{\beta}, \boldsymbol{\theta} = \boldsymbol{\phi}, \sigma_u^2 = \sigma_v^2} L(\mathbf{x}, \mathbf{y}; \boldsymbol{\alpha}, \boldsymbol{\theta}, \boldsymbol{\beta}, \boldsymbol{\phi}, \sigma_u^2, \sigma_v^2)}{\max\limits_{\boldsymbol{\alpha}, \boldsymbol{\theta}, \boldsymbol{\beta}, \boldsymbol{\phi}, \sigma_u^2, \sigma_v^2} L(\mathbf{x}, \mathbf{y}; \boldsymbol{\alpha}, \boldsymbol{\theta}, \boldsymbol{\beta}, \boldsymbol{\phi}, \sigma_u^2, \sigma_v^2)} . \tag{31}$$

Taking $-2 \cdot \log\ T$ yields

$$
\begin{aligned}
-2 \cdot \log T &= (nx+ny) \cdot \log \tilde{\sigma}^2 + \sum_{t=1}^{nx} \log \tilde{a}_t + \sum_{t=1}^{ny} \log \tilde{b}_t \\
&\quad -nx \cdot \log \hat{\sigma}_x^2 - \sum_{t=1}^{nx} \log \hat{a}_t - ny \cdot \log \hat{\sigma}_y^2 - \sum_{t=1}^{ny} \log \hat{b}_t \ , \qquad (32)
\end{aligned}
$$

where $\tilde{\sigma}^2$, $\tilde{a}_t$, and $\tilde{b}_t$ are obtained by maximizing the numerator of equation (31) and $\hat{\sigma}_x^2$, $\hat{\sigma}_y^2$, $\hat{a}_t$, and $\hat{b}_t$ are obtained by maximizing its denominator. For this case, $-2 \cdot \log T$ has an asymptotic chi-square distribution with $p + q + 1$ degrees of freedom.

3.2 TEST FOR EQUAL ARMA PARAMETERS, INNOVATIONS VARIANCES, AND PROCESS MEANS

The test in equation (32) assumed zero mean or equal mean data. Here, the test for $\mu_x = \mu_y$ is included in the likelihood ratio test:

$$
T = \frac{\max\limits_{\boldsymbol{\alpha}=\boldsymbol{\beta},\ \boldsymbol{\theta}=\boldsymbol{\phi},\sigma_u^2=\sigma_v^2,\ \mu_x=\mu_y} L(\mathbf{x},\mathbf{y};\boldsymbol{\alpha},\boldsymbol{\theta},\boldsymbol{\beta},\boldsymbol{\phi},\sigma_u^2,\sigma_v^2,\mu_x,\mu_y)}{\max\limits_{\boldsymbol{\alpha},\ \boldsymbol{\theta},\ \boldsymbol{\beta},\ \boldsymbol{\phi},\sigma_u^2,\ \sigma_v^2,\ \mu_x,\ \mu_y} L(\mathbf{x},\mathbf{y};\boldsymbol{\alpha},\boldsymbol{\theta},\boldsymbol{\beta},\boldsymbol{\phi},\sigma_u^2,\sigma_v^2,\mu_x,\mu_y)} . \qquad (33)
$$

Taking $-2 \cdot \log T$ yields equation (32), where $\tilde{\sigma}^2$, $\tilde{a}_t$, and $\tilde{b}_t$ are obtained by maximizing the numerator of equation (33) and $\hat{\sigma}_x^2$, $\hat{\sigma}_y^2$, $\hat{a}_t$, and $\hat{b}_t$ are obtained by maximizing its denominator. Here, $-2 \cdot \log T$ has an asymptotic chi-square distribution with $p + q + 2$ degrees of freedom.

3.3 TEST FOR EQUAL ARMA PARAMETERS

In many applications, only the ARMA structure of the processes is of interest. The test for $\boldsymbol{\alpha} = \boldsymbol{\beta}$ and $\boldsymbol{\theta} = \boldsymbol{\phi}$ is given by

$$
T = \frac{\max\limits_{\boldsymbol{\alpha}=\boldsymbol{\beta},\ \boldsymbol{\theta}=\boldsymbol{\phi},\sigma_u^2,\ \sigma_v^2,\ \mu_x,\ \mu_y} L(\mathbf{x},\mathbf{y};\boldsymbol{\alpha},\boldsymbol{\theta},\boldsymbol{\beta},\boldsymbol{\phi},\sigma_u^2,\sigma_v^2,\mu_x,\mu_y)}{\max\limits_{\boldsymbol{\alpha},\ \boldsymbol{\theta},\ \boldsymbol{\beta},\ \boldsymbol{\phi},\sigma_u^2,\ \sigma_v^2,\ \mu_x,\ \mu_y} L(\mathbf{x},\mathbf{y};\boldsymbol{\alpha},\boldsymbol{\theta},\boldsymbol{\beta},\boldsymbol{\phi},\sigma_u^2,\sigma_v^2,\mu_x,\mu_y)} . \qquad (34)
$$

The sample means $\bar{x}$ and $\bar{y}$ are subtracted from the data $\mathbf{x}$ and $\mathbf{y}$, respectively. Substituting equation (29) into the numerator and denominator of equation (34) and then maximizing the numerator and denominator yields the test value of the statistic T. Taking $-2 \cdot \log T$ yields

$$\begin{aligned} -2 \cdot \log T &= nx \cdot \log \tilde{\sigma}_x^2 + \sum_{t=1}^{nx} \log \tilde{a}_t \\ &\quad + ny \cdot \log \tilde{\sigma}_y^2 + \sum_{t=1}^{ny} \log \tilde{b}_t \\ &\quad - nx \cdot \log \hat{\sigma}_x^2 - \sum_{t=1}^{nx} \log \hat{a}_t \\ &\quad - ny \cdot \log \hat{\sigma}_y^2 - \sum_{t=1}^{ny} \log \hat{b}_t \ , \end{aligned} \tag{35}$$

where $\tilde{\sigma}_x^2$, $\tilde{\sigma}_y^2$, $\tilde{a}_t$, and $\tilde{b}_t$ are obtained from maximizing the numerator and $\hat{\sigma}^2$, $\hat{\sigma}^2$, $\hat{a}_t$, and $\hat{b}_t$ are obtained from maximizing the denominator. For this case, $-2 \cdot \log T$ is asymptotically chi-square distributed with $p + q$ degrees of freedom.

3.4 TEST FOR EQUAL ARMA PARAMETERS AND PROCESS MEANS

In this special case, there is a trend in the data. The likelihood ratio test for $\boldsymbol{\alpha} = \boldsymbol{\beta}$, $\boldsymbol{\theta} = \boldsymbol{\phi}$, and $\mu_x = \mu_y$ is

$$T = \frac{\displaystyle\max_{\boldsymbol{\alpha}=\boldsymbol{\beta},\ \boldsymbol{\theta}=\boldsymbol{\phi},\sigma_u^2,\ \sigma_v^2,\ \mu_x=\mu_y} L(\mathbf{x}-\bar{m}, \mathbf{y}-\bar{m}; \boldsymbol{\alpha}, \boldsymbol{\theta}, \boldsymbol{\beta}, \boldsymbol{\phi}, \sigma_u^2, \sigma_v^2, \mu_x, \mu_y)}{\displaystyle\max_{\boldsymbol{\alpha},\ \boldsymbol{\theta},\ \boldsymbol{\beta},\ \boldsymbol{\phi},\sigma_u^2,\ \sigma_v^2,\ \mu_x,\ \mu_y} L(\mathbf{x}-\bar{x}, \mathbf{y}-\bar{y}; \boldsymbol{\alpha}, \boldsymbol{\theta}, \boldsymbol{\beta}, \boldsymbol{\phi}, \sigma_u^2, \sigma_v^2, \mu_x, \mu_y)} \ , \tag{36}$$

where $\bar{x} = (1/nx) \sum_{t=1}^{nx} x_t$, $\bar{y} = (1/ny) \sum_{t=1}^{ny} y_t$, and $\bar{m} = (1/(nx+ny)) \left[\sum_{t=1}^{nx} x_t + \sum_t^{ny} y_t\right]$. Maximizing the numerator and denominator with equation (29) and then taking $-2 \cdot \log T$ yields

$$-2 \cdot \log T = nx \cdot \log \tilde{\sigma}_x^2 + \sum_{t=1}^{nx} \log \tilde{a}_t$$

$$
\begin{aligned}
&+ny \cdot \log \tilde{\sigma}_y^2 + \sum_{t=1}^{ny} \log \tilde{b}_t \\
&-nx \cdot \log \hat{\sigma}_x^2 - \sum_{t=1}^{nx} \log \hat{a}_t \\
&-ny \cdot \log \hat{\sigma}_y^2 - \sum_{t=1}^{ny} \log \hat{b}_t \ ,
\end{aligned}
\tag{37}
$$

where $\tilde{\sigma}_x^2$, $\tilde{\sigma}_y^2$, $\tilde{a}_t$, and $\tilde{b}_t$ are obtained by maximizing the numerator of equation (36) and $\hat{\sigma}_x^2$, $\hat{\sigma}_y^2$, $\hat{a}_t$, and $\hat{b}_t$ are obtained by maximizing its denominator. Here $-2 \cdot \log T$ is asymptotically chi-square distributed with $p + q + 1$ degrees of freedom.

4. MONTE CARLO ANALYSIS

In this section, the time series test is examined to see how well it fits the chi-square distribution. As an example, the test of equation (16) is applied to two AR(4) time series, both defined by

$$x_t = 2.0625x_{t-1} - 2.4325x_{t-2} + 1.5847x_{t-3} - 0.652x_{t-4} + u_t \ . \tag{38}$$

Two independent realizations of the above process were simulated. The AR model order was chosen using the AIC selection procedure. The value of the model order, p, used in the test was selected from the pooled process. The test statistic was calculated for 10,000 pairs of realizations with a sample size of 200. This statistic was then compared against a chi-square distribution with $(p + 1)$ degrees of freedom. As a function of model order, table 1 shows the fraction of the test statistic values that exceeded the chi-square percentage points for the significance level indicated in the top row. These results indicate that the test has a good fit to the chi-square distribution. However, the test is, in general, conservative, which means that the test rejects the null hypothesis less often than predicted by the chi-square distribution. Depending on the application, a conservative test may be more desirable than a liberal test because it has fewer false alarms.

Table 1. Fraction of the Test Statistic Values Exceeding the Chi-Square Percentage Point for Equal Models

Order	Samples	0.5	0.25	0.1	0.05	0.025	0.01
4	5642	0.4975	0.2407	0.0939	0.0473	0.0245	0.0105
5	1559	0.4894	0.2450	0.0943	0.0500	0.0218	0.0051
6	1033	0.5073	0.2536	0.1162	0.0610	0.0358	0.0181
7	845	0.4852	0.2379	0.0959	0.0367	0.0107	0.0047
8	921	0.5005	0.2617	0.0966	0.0445	0.0206	0.0087
combined	10000	0.4965	0.2444	0.9670	0.0480	0.0237	0.0098

The previous paragraph analyzes the test statistic when the null hypothesis is true. Now the power of the test is examined when the AR processes are different. One of the processes is as defined above for x_t; the second is defined by

$$y_t = 1.971y_{t-1} - 2.339y_{t-2} + 1.5208y_{t-3} - 0.6544y_{t-4} + v_t \ .$$

The simulation procedure described above was applied to these two processes, with the results presented in table 2. At the 0.05-significance level, the test rejects the equality of the model parameters of the two processes for 15 to 20 percent of the realizations, as compared to 5 percent when the two processes are the same. These results are encouraging, considering the "closeness" of the two processes, which is measured by the location of the zeros for the characteristic equation of the AR process. Table 3 shows the characteristic equation zeros for the two processes. A larger separation between the zeros than shown in this table will result in a greater rejection of equality.

Table 2. Fraction of the Test Statistic Values Exceeding the Chi-Square Percentage Point for Unequal Models

Order	Samples	0.5	0.25	0.1	0.05	0.025	0.01
4	5538	0.7483	0.5256	0.3044	0.1956	0.1201	0.0634
5	1572	0.7366	0.5115	0.3015	0.1915	0.1126	0.0630
6	1024	0.7119	0.4932	0.2734	0.1787	0.1133	0.0684
7	868	0.7051	0.4493	0.2385	0.1336	0.0680	0.0323
8	998	0.7074	0.4689	0.2485	0.1483	0.0882	0.0451
combined	10000	0.7349	0.5078	0.2895	0.1831	0.1105	0.0593

Table 3. The Coefficients and Zeros of the Two AR(4) Models

Series 1		Series 2
	Coefficients	
2.0624611		1.9710257
-2.43250		-2.3390543
1.5846875		1.5208008
-0.6520562		-0.6543516
	Zeros (polar)	
(0.950 , ±0.100)		(0.945 , ±0.105)
(0.850 , ±0.200)		(0.856 , ±0.205)

5. CONCLUSIONS AND RECOMMENDATIONS

The analysis in this report has studied the equality of the parameters for two ARMA processes. Four likelihood ratio tests were derived to compare (a) the ARMA parameters and innovations variances, (b) the ARMA parameters, innovations variances, and process means, (c) the ARMA parameters, and (d) the ARMA parameters and process means. The likelihood ratio test statistic has been shown to have a good fit to the chi-square distribution when the models are identical.

The test statistics presented here are examples of the generalized likelihood ratio test

$$T = \frac{\max\limits_{\boldsymbol{\theta}_x = \boldsymbol{\theta}_y} L(\mathbf{x},\ \mathbf{y};\ \boldsymbol{\theta}_x,\ \boldsymbol{\theta}_y)}{\max\limits_{\boldsymbol{\theta}_x\ ,\ \boldsymbol{\theta}_y} L(\mathbf{x},\ \mathbf{y};\ \boldsymbol{\theta}_x,\ \boldsymbol{\theta}_y)} ,$$

where $L(\cdot)$ is the likelihood of the data and $\boldsymbol{\theta}_x$ and $\boldsymbol{\theta}_y$ are the p-vectors of the model parameters that characterize the data. As shown in the report, $-2 \cdot \log T$ has an asymptotic chi-square distribution with p degrees of freedom when the null hypothesis ($\boldsymbol{\theta}_x = \boldsymbol{\theta}_y$) is true.

Three recommendations for further research are (1) to extend the test to complex-valued processes, (2) to study the sensitivity of the power of the test when the two process models become close, as measured by the zeros of the characteristic equation, and (3) to expand the test for three or more time series.

For future sonar systems, these test procedures can be utilized in the automatic detection and classification of sonar signals, which will reduce the workload and improve the performance of the sonar operator.

APPENDIX

LOG LIKELIHOOD EVALUATION FOR THE ARMA MODEL

This appendix presents a Kalman filter[4–6] procedure for evaluating the log likelihood of $L(x; \alpha, \theta, \sigma_u^2, \mu_x)$ (or log $L(y; \beta, \phi, \sigma_v^2, \mu_y)$), given the model parameters. For the Kalman filter, let the observation x_t be given by the *observation equation*

$$x_t = z^T w_t + r_t \ , \tag{A-1}$$

where x_t is the observed data, z is an m x 1 vector of fixed known values, w_t is the state vector of the system, and r_t is the measurement error.

The *state equation* of the system is given by

$$w_t = Aw_{t-1} + Be_t \ ,$$

where A is an m x m *transition matrix* of fixed known values, B is an m x k matrix of fixed known values, and e_t is a k x 1 vector of Gaussian random variables with

$$\begin{aligned} E[e_t] &= 0 \ , \\ E[e_t e_t^T] &= \sigma^2 Q \ , \\ E[e_t e_s^T] &= 0 \text{ for } t \neq s \ . \end{aligned}$$

The known matrix Q is positive definite. Further,

$$\begin{aligned} E[r_t] &= 0 \ , \\ E[r_t^2] &= R \ , \\ E[r_t r_s] &= 0 \text{ for } s \neq t \ , \\ E[r_t e_s] &= 0 \text{ for all } s \text{ and } t \ . \end{aligned}$$

Let $w(t-1 \mid t-1)$ denote the minimum mean-square estimate of w_{t-1} given the measurements $x_1, x_2, \ldots , x_{t-1}$. Let $\sigma^2 P(t-1 \mid t-1)$ denote the estimation error covariance matrix where $P(t-1 \mid t-1)$ is known. That is,

$$E[(w_{t-1} - w(t-1 \mid t-1))(w_{t-1} - w(t-1 \mid t-1))^T] = \sigma^2 P(t-1 \mid t-1) \ .$$

Let $w(t \mid t-1)$ and $P(t \mid t-1)$ denote the predicted values of w_t and $P(t \mid t)$ given x_1, x_2, ... , x_{t-1}. These quantities are given by the *prediction equations*

$$w(t \mid t-1) = Aw(t-1 \mid t-1), \tag{A-2}$$

$$P(t \mid t-1) = AP(t-1 \mid t-1)A^T + BQB^T. \tag{A-3}$$

Using the observation x_t, $w(t \mid t)$ and $P(t \mid t)$ are obtained by the *update equations*

$$w(t \mid t) = w(t \mid t-1) + P(t \mid t-1)za_t^{-1}(x_t - z^T w(t \mid t-1)) \ , \tag{A-4}$$

$$P(t \mid t) = P(t \mid t-1) - P(t \mid t-1)za_t^{-1}z^T P(t \mid t-1) \ , \tag{A-5}$$

where $a_t = z^T P(t \mid t-1)z + R$.

The ARMA(p,q) model for x_t, given in equation (27), can be written as

$$x_t = \sum_{k=1}^{r} \alpha_k x_{t-k} + \sum_{k=1}^{r} \theta_k u_{t-k} + u_t \ , \tag{A-6}$$

where $r = \max(p,q+1)$. Alternatively, equation (A-6) can be expressed in Markovian form by

$$w_t = \begin{bmatrix} \alpha_1 & 1 & 0 & . & . & . & 0 \\ \alpha_2 & 0 & 1 & . & . & . & 0 \\ . & . & & & & . & . \\ . & . & & & & . & . \\ . & . & & & & . & . \\ \alpha_{r-1} & 0 & . & . & . & 0 & 1 \\ \alpha_r & 0 & . & . & . & 0 & 0 \end{bmatrix} w_{t-1} + \begin{bmatrix} 1 \\ \theta_1 \\ \theta_2 \\ . \\ . \\ . \\ \theta_{r-1} \end{bmatrix} e_t \ , \tag{A-7}$$

where the first element of w_t is x_t. Equation (A-7) is the state equation in the state space formulation of the ARMA(p,q) model. The *measurement equation* is $x_t = z^T w_t$, where $z = (1, 0, \dots ,0_r)^T$.

For the ARMA(p,q) model without measurement error, $R = 0$, $Q = 1$,

$$A = \begin{bmatrix} \alpha_1 & 1 & 0 & . & . & . & 0 \\ \alpha_2 & 0 & 1 & . & . & . & 0 \\ . & . & & & & . & . \\ . & . & & & & . & . \\ . & . & & & & . & . \\ \alpha_{r-1} & 0 & . & . & . & 0 & 1 \\ \alpha_r & 0 & . & . & . & 0 & 0 \end{bmatrix},$$

and

$$B = \begin{bmatrix} 1 \\ \theta_1 \\ \theta_2 \\ . \\ . \\ . \\ \theta_{r-1} \end{bmatrix}.$$

To start the recursions in equations (A-2) through (A-5), initial values are needed for $w(0 \mid 0)$ and $P(0 \mid 0)$. Without any measurements, the minimum mean-square estimate of w_0 is zero; hence, $w(0 \mid 0)$ is set to zero. Now, since $P(0 \mid 0) = [w_0 w_0^T]$, the covariance matrix of the state vector, the value of $P(0 \mid 0)$ is the solution P of the equation

$$P = APA^T + BQB^T \ . \tag{A-8}$$

Using equation (A-2) and noting the stationarity assumptions on the model allows equation (A-8) to be expressed as

$$[I - A \otimes A] vec(P) = vec(BQB^T), \tag{A-9}$$

where $\otimes$ is the Kronecker product. To solve this system of r^2 equations, the r^2 x r^2 matrix $[I - A \otimes A]$ does not require inversion. Rather, the columns of P can solved recursively using the following relations:

$$p_1 = \left(I - \sum_{k=1}^{r} \alpha_k A^k \right)^{-1} \left(\sum_{k=0}^{r-1} \beta_k A^k \right) B \tag{A-10}$$

and

$$p_i = \alpha_i A p_1 + A p_{i+1} + \beta_{i-1} B \text{ for } i > 1 \ . \tag{A-11}$$

After solving for $P_0 = P$, the log likelihood can be evaluated using equations (A-2) through (A-5), given the ARMA(p,q) model parameters.

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